Examination in General Relativity

August 29, 2018, 8:00 – 13:00

Please make sure that your answers are easy to read and have enough details to be followed. Calculations or reasoning should always be provided, unless otherwise stated in the problem. You are free to use formulas and expressions from the course book, but if you do, be careful to state its equation number, and explain why it is relevant to the solution.

Allowed help: Pocket calculator and course book (Hartle: Gravity).

Maximum points: 24 p. A passing grade requires at least 12 p.

1.

(a) Consider two observers, A and B, standing close to each other on the surface of the earth. Observer A is just standing still with his clock. Observer B throws her clock straight up. It returns after time T on A:s clock. What time has then elapsed on B:s clock? Assume that the height of the trajectory is much smaller than the radius of the earth (so that the gravitational acceleration g can be taken to be constant), neglect air resistance (so that clock B is in free fall), assume the trajectory is the Newtonian trajectory (that is, just constant acceleration with respect to the surface of the earth), and calculate only to order $1/c^2$. (4 p)

(b) A third observer C has attached his clock to a drone, and lets it travel straight up to the same height as clock B, and then back again, but with constant speed, so that it reaches the ground after the same time T on A:s clock. Compared to the time on clock B, will C show a shorter or longer time? No calculation is required in this part, just an answer with a short explanation. (1 p)

[Modelled on Hartle problem 6.13]

2. Consider the 2-dimensional spacetime spanned by coordinates (v, x) with the line element

$$ds^2 = -x\,dv^2 + 2\,dv\,dx$$

(a) Calculate the light cone at point (v, x), that is, find the slopes of the two light rays through an arbitrary point (v, x)! (2 p)

(b) Sketch the light cones in a (v, x)-diagram along a v = constant line. Make clear what is "the inside" of the light cones, that is, what are the timelike directions, and why. What line x = constant can only be crossed in one direction? (3 p)

[Modelled on Hartle problem 7:5]

3. Suppose that we have a metric that in a certain coordinate system does not depend on one of the coordinates, say x_1. Then there is a Killing field ξ which in this coordinate system takes the form

$$\xi^\alpha = (0, 1, 0, 0)$$

Show by explicit calculation that the Killing field then satisfies Killing's equation:

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$$

(4 p)

[Modelled on Hartle problem 20:18]

4. Two particles fall in radially from infinity in the Schwarzschild geometry. One starts with $e = 1$, and the other with $e = 2$ (where e is the energy per unit mass as measured at infinity). A stationary observer at $r = 6M$ measures the speed of each particle when they pass by. How much faster is the second particle moving at that point according to the observer? (5 p)

   [Modelled on Hartle problem 9:7]

5. Consider a closed FRW model (that is, $k = +1$) containing a matter density ρ_m, a vacuum energy ρ_Λ corresponding to a positive cosmological constant Λ, but no radiation. In answering the questions below, start from the Friedmann equation

   $$\dot{a}^2 - \frac{8\pi\rho}{3}a^2 = -k$$

   and the scaling of the matter density and vacuum energy, respectively:

   $$\rho_m = \frac{\rho_{m,0}a_0^3}{a^3} \qquad \rho_\Lambda = \frac{\Lambda}{8\pi}$$

   (a) Show that for given Λ there is a critical value of ρ_m for which the scale factor a is independent of time – that is, the universe is static. Find this critical value of ρ_m. (3 p)

   (b) What is the spatial volume of this universe, expressed in terms of Λ? (2 p)

   [Modelled on Hartle problem 18:24]