IMPORTANT SPACETIMES (geometrized units)

Flat Spacetime

Cartesian Coordinates

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \equiv \eta_{\alpha\beta} dx^\alpha dx^\beta$$

Spatial Spherical Polar Coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Static, Weak Field Metric

$$ds^2 = -(1 + 2\Phi(x^i))\, dt^2 + (1 - 2\Phi(x^i))(dx^2 + dy^2 + dz^2), \quad (\Phi(x^i) \ll 1).$$

Schwarzschild Geometry

Schwarzschild Coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Eddington–Finkelstein Coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Kruskal–Szekeres Coordinates

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dV^2 + dU^2\right) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Kerr Geometry

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$
$$+ \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right) \sin^2\theta\, d\phi^2,$$

where

$$a \equiv J/M, \quad \rho^2 \equiv r^2 + a^2\cos^2\theta, \quad \Delta \equiv r^2 - 2Mr + a^2$$

Linearized Plane Gravitational Wave

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + h_{\alpha\beta} dx^\alpha dx^\beta$$

where (rows and columns in t, x, y, z order)

$$h_{\alpha\beta}(t, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_\times(t-z) & 0 \\ 0 & f_\times(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for a wave propagating in the z-direction.

Friedman–Robertson–Walker Cosmological Models

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \left\{\begin{matrix} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{matrix}\right\}(d\theta^2 + \sin^2\theta \, d\phi^2)\right], \quad \left\{\begin{matrix} \text{closed} \\ \text{flat} \\ \text{open} \end{matrix}\right\}.$$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)\right], \quad \begin{pmatrix} k = +1, \text{ closed} \\ k = 0, \text{ flat} \\ k = -1, \text{ open} \end{pmatrix}.$$

THE GEODESIC EQUATION

- Lagrangian for the Geodesic Equation of a test particle

$$L\left(\frac{dx^\alpha}{d\sigma}, x^\alpha\right) = \left(-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right)^{1/2}$$

where σ is an arbitrary parameter along the world line $x^\alpha = x^\alpha(\sigma)$ of the geodesic.

- Geodesic equation for a test particle (coordinate basis)

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \quad \text{or} \quad \frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$$

where τ is the proper time along the geodesic and $u^\alpha = dx^\alpha/d\tau$ are the coordinate basis components of the four-velocity so that $\mathbf{u} \cdot \mathbf{u} = -1$. The Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ follow from Lagrange's equations or from the general formula (8.19). The geodesic equation for light rays takes the same form with τ replaced by an affine parameter and $\mathbf{u} \cdot \mathbf{u} = 0$.

- Conserved Quantities

$$\boldsymbol{\xi} \cdot \mathbf{u} = \text{constant}$$

where $\boldsymbol{\xi}$ is a Killing vector, e.g., $\xi^\alpha = (0, 1, 0, 0)$ in a coordinate basis where the metric $g_{\alpha\beta}(x)$ is independent of x^1.

COORDINATE AND ORTHONORMAL BASES

- A set $\{\mathbf{e}_{\hat{\alpha}}\}$ of four *orthonormal* basis vectors satisfies

$$\mathbf{e}_{\hat{\alpha}}(x) \cdot \mathbf{e}_{\hat{\beta}}(x) = \eta_{\hat{\alpha}\hat{\beta}}.$$

- A set $\{\mathbf{e}_{\alpha}\}$ of four *coordinate* basis vectors associated with a set of coordinates x^{α} satisfies

$$\mathbf{e}_{\alpha}(x) \cdot \mathbf{e}_{\beta}(x) = g_{\alpha\beta}(x)$$

where the line element has the form $ds^2 = g_{\alpha\beta}(x)dx^{\alpha}dx^{\beta}$.

- If the coordinate system is *orthogonal* ($g_{\alpha\beta}(x) = 0$ for $\alpha \neq \beta$), the coordinate basis components of an orthonormal basis pointing along the coordinate directions have the form

$$(\mathbf{e}_{\hat{0}})^{\alpha} = [(-g_{00})^{-1/2}, 0, 0, 0], \quad (\mathbf{e}_{\hat{1}})^{\alpha} = [0, (g_{11})^{-1/2}, 0, 0], \quad \text{etc.}$$

USEFUL NUMBERS

Conversion Factors

Velocity of light	$c \equiv 299792458$ m/s $\approx 3 \times 10^{10}$ cm/s
Boltzmann's constant	$k_B = 1.38 \times 10^{-16}$ erg/K $= 8.59 \times 10^{-5}$ eV/K
Second of arc	1 arcsec = 1″ =4.85×10^{-6} rad
Light year	1 ly = 9.46×10^{17} cm
Parsec	1 pc = 3.09×10^{18} cm = 3.26 ly
Electron volt	1 eV = 1.60×10^{-12} erg = 1.16×10^{4} K
Erg (cgs unit of energy)	1 erg = 10^{-7} J
Dyne (cgs unit of force)	1 dyne = 10^{-5} N

Physical Constants

Gravitational constant	$G = 6.67 \times 10^{-8}$ dyn $\cdot$ cm^2/g^2
Stefan–Boltzmann constant	$\sigma = 5.67 \times 10^{-5}$ erg/(cm$^2 \cdot$ s $\cdot$ K^4)
Radiation constant	$a = 7.56 \times 10^{-15}$ erg/(cm$^3 \cdot$ K^4)
Mass of an electron	$m_e = 9.11 \times 10^{-28}$ g
Mass of a proton	$m_p = 1.67 \times 10^{-24}$ g
Planck's constant	$\hbar = 1.05 \times 10^{-27}$ erg $\cdot$ s

ASTRONOMICAL CONSTANTS

Earth

Astronomical unit	$AU = 1.50 \times 10^8$ km
(semimajor axis of Earth's orbit)	$= 1.50 \times 10^{13}$ cm
Mass of the Earth	$M_\oplus = 5.97 \times 10^{27}$ g
	$GM_\oplus/c^2 = 0.443$ cm
Equatorial radius of the Earth	$R_\oplus = 6.38 \times 10^8$ cm $= 6378$ km
Moment of inertia about rotation axis	8.04×10^{44} g $\cdot$ cm$^2 = .331\ M_\oplus R_\oplus^2$
Rotation period	8.62×10^4 s
Angular velocity	$\Omega_\oplus = 7.29 \times 10^{-5}$ rad/s

Sun

Mass of the Sun	$M_\odot = 1.99 \times 10^{33}$ g
	$GM_\odot/c^2 = 1.48$ km
Radius of the Sun	$R_\odot = 6.96 \times 10^{10}$ cm $= 6.96 \times 10^5$ km
Moment of inertia about rotation axis	5.7×10^{53} g $\cdot$ cm^2
Rotation period at Equator	25.5 days
Angular velocity at Equator	2.85×10^{-6} rad/s
Luminosity of the Sun	$L_\odot = 3.85 \times 10^{33}$ erg/s

Moon

Radius of the Moon's orbit (mean)	3.84×10^5 km
Mass of the Moon	$M_{\text{Moon}} = 7.35 \times 10^{25}$ g $= M_\oplus/81.3$
Radius of the Moon	$R_{\text{Moon}} = 1.74 \times 10^3$ km

Our Galaxy (The Milky Way)

Mass of the Milky Way in visible matter	$\approx 10^{11} M_\odot$
Radius of the luminous Milky Way disk	$\approx 20 - 25$ kpc
Luminosity of the Milky Way	$\approx 4 \times 10^{10}\ L_\odot$

Universe

Hubble Constant	$H_0 \approx (72 \pm 7)[(\text{km/s})/\text{Mpc}]$
	$h \equiv H_0/(100\ [(\text{km/s})/\text{Mpc}]) \approx .7 \pm .1$
Hubble Time	$t_H \equiv H_0^{-1} = 9.78 \times 10^9\ h^{-1}$ yr
Hubble Distance	$d_H \equiv cH_0^{-1} = 2998\ h^{-1}$ Mpc
Critical density	$\rho_c \equiv 3H_0^2/8\pi G = 1.88 \times 10^{-29}\ h^2$ g/cm^3
Temperature of CMB today	$= 2.73$K

APPENDIX A

Units

A.1 Units in General

To understand something of units, imagine the problem of communicating the predictions of our physical theories to intelligent aliens living near a distant star. You can send a message saying that mass of the proton is approximately 1835 times the mass of the electron; that ratio of masses is a dimensionless number that can be transmitted as bits. But a message saying that mass of a proton is 1.672×10^{-27} kg will make no sense. The aliens don't know what a kilogram is. Nor can we explain it to them exactly because it is defined as the mass of the block of metal kept in the Bureau International des Poids et Mesures, in Sèvres outside Paris. You could send a message saying that the standard kilogram was *approximately* 5.980×10^{26} proton masses because that is a dimensionless ratio between the mass of the international kilogram and the proton mass. That will be a less interesting message because it is not about the predictions of physical laws, but rather about how humans organize those predictions.

The predictions of fundamental physical theories are reducible to dimensionless numbers. Units are introduced for *convenience*, and the number and system of units varies considerably with the notion of convenience. For example, today the second is *defined* as the time required for exactly 9,192,631,770 cycles in the transition between the two lowest energy states of a cesium atom, and the meter is *defined* to be $1/(299{,}792{,}458)$ of one of those seconds—both definitions involving defined dimensionless numbers. We use hours, minutes, and seconds partly because of tradition, but also because it would be inconvenient to talk about lectures that were 28 trillion cycles of a cesium transition long. Were it convenient, we *could* introduce a unit to measure the areas of circles that differs from that used to measure the area of rectangles. Suppose a 1-cm radius circle is defined to have an area of 1 Archimedes. Then there would be the conversion 1 Archimedes equals 3.14159265... cm^2. That would add one more unit but not much convenience, so there is little motivation for the Archimedes. But from the prespective of special relativity the use of separate units to measure spacelike and timelike distances is not so very different (Section 4.6). When a dimensionless ratio is between a measured quantity and a *standard*, then it is convenient to introduce a unit for the standard as in the case of the international kilogram.

Accepted physical theory plays an important role in the choice of units. The second could not be defined as above if we did not have confidence from the many successes of atomic theory that all cesium atoms were identical. Confidence

A

T
pr
Y
ti
th
1.
is
bl
ou
ap
be
a l
bu

le
of
th
tra
de
in
be
tu
we
us
ha
eq
nie
of
dis
tw
for

sec
ma

in special relativity is behind the definition of the meter in terms of time units because that theory asserts that the velocity of light is the same in all intertial frames.

Progress in experiment also plays a role in determining what units are used. One reason we have separate units for mass, length, and time is that there once were separate standards for these quantities. The second was once defined as a certain fraction of the mean solar day, and the meter by the distance between two marks on a particular bar. When it became possible to measure the frequency of atomic transitions more accurately than the solar day it made sense to change the definition of the unit of time to the one we have today. Future progress could change the present situation. For example, with confidence in the equality of gravitational and intertial mass, general relativity, and access to precise enough measurements, the kilogram could be defined as the mass of a sphere such that a test mass completes a circular orbit of radius 1 m in some defined number of days. (At current accuracies that would be approximately 8.90 days from Kepler's law (3.24)). Newton's gravitational constant would then be a defined quantity, rather than a measured one, just like the velocity of light is today. Indeed by defining mass as an appropriate multiple of the inverse period squared of such an orbit G could be made equal to 1.

A.2 Units Employed in This Book

This text employs three systems of units for mechanics and the special and general theories of relativity that are convenient in different circumstances. For the traditional mass-length-time ($\mathcal{MLT}$) system we use the (cgs) units of gram, centimeter, and second. These are standard in astrophysics for most of the applications we consider. The units convenient for special relativity are a mass-length ($\mathcal{ML}$) system, in which the velocity of light is unity ($c = 1$). The gram and the centimeter are used for these units. The units convenient for general relativity are a length ($\mathcal{L}$) system called *geometrized units*, in which $G = 1$ and $c = 1$ where mass, length, and time all have units of length.

Tables A.1 and A.2 show how to convert various quantities between $\mathcal{MLT}$ units, $\mathcal{ML}$ ($c = 1$) units, and $\mathcal{L}$ ($G = c = 1$) units. The tables can be used in two ways: To convert from $\mathcal{MLT}$ units to either of the other systems, multiply by the indicated factor in the last column. For instance, to convert mass in grams to mass in centimeters use the first line of Table A.2 to find

$$M(\text{in cm}) = (G/c^2)M(\text{in g}) = .742 \times 10^{-28} M(\text{in g}). \quad (\text{A.1})$$

To convert equations back to $\mathcal{MLT}$ units from either of the other two systems, replace quantities by the expressions in the last column with c and G restored. For instance, the equation giving the escape velocity of a particle from a Schwarzschild coordinate radius R outside a spherical black hole of mass M is [cf. (9.42)] $V_{\text{escape}} = (2M/R)^{1/2}$ in geometrized units. To find the same relation in $\mathcal{MLT}$ units we find from Table A.1 that V_{escape} should be replaced by V_{escape}/c and from ible A.2 that M should be replaced by GM/c^2. This gives

$$\frac{V_{\text{escape}}}{c} = \left(\frac{2GM}{c^2R}\right)^{1/2}, \quad \text{or} \quad V_{\text{escape}} = \left(\frac{2GM}{R}\right)^{1/2}. \tag{A.2}$$

ιBLE A.1 Mass-Length and Mass-Length-Time Units

Ꝺuantity	Typical symbol	$\mathcal{ML}$ unit	$\mathcal{MLT}$ unit	Conversion $\mathcal{MLT} \to \mathcal{ML}$
Mass	m	$\mathcal{M}$	$\mathcal{M}$	m
Length	L	$\mathcal{L}$	$\mathcal{L}$	L
Time	t	$\mathcal{L}$	$\mathcal{T}$	ct
ͻacetime listance	s	$\mathcal{L}$	$\mathcal{L}$	s
ͻper time	τ	$\mathcal{L}$	$\mathcal{T}$	$c\tau$
Energy	E	$\mathcal{M}$	$\mathcal{M}(\mathcal{L}/\mathcal{T})^2$	E/c^2
ͻmentum	p	$\mathcal{M}$	$\mathcal{M}(\mathcal{L}/\mathcal{T})$	p/c
′elocity	V	dimensionless	$\mathcal{L}/\mathcal{T}$	V/c

.BLE A.2 Geometrized and Mass-Length-Time Units

Quantity	Typical symbol	Geometrized unit	$\mathcal{MLT}$ unit	Conversion $\mathcal{MLT} \to$ geom.
Mass	M	$\mathcal{L}$	$\mathcal{M}$	GM/c^2
Length	L	$\mathcal{L}$	$\mathcal{L}$	L
Time	t	$\mathcal{L}$	$\mathcal{T}$	ct
Spacetime distance	s	$\mathcal{L}$	$\mathcal{L}$	s
Proper time	τ	$\mathcal{L}$	$\mathcal{T}$	$c\tau$
Energy	E	$\mathcal{L}$	$\mathcal{M}(\mathcal{L}/\mathcal{T})^2$	GE/c^4
Momentum	p	$\mathcal{L}$	$\mathcal{M}(\mathcal{L}/\mathcal{T})$	Gp/c^3
Angular momentum	J	$\mathcal{L}^2$	$\mathcal{M}(\mathcal{L}^2/\mathcal{T})$	GJ/c^3
Power (luminosity)	L	dimensionless	$\mathcal{ML}^2/\mathcal{T}^3$	GL/c^5
Energy density	ϵ	$\mathcal{L}^{-2}$	$\mathcal{M}/(\mathcal{LT}^2)$	$G\epsilon/c^4$
Momentum density (energy flux)	$\vec{\pi}$	$\mathcal{L}^{-2}$	$\mathcal{M}/(\mathcal{L}^2\mathcal{T})$	$G\vec{\pi}/c^3$
Pressure (stress)	p	$\mathcal{L}^{-2}$	$\mathcal{M}/(\mathcal{LT}^2)$	Gp/c^4
Energy of an orbit per unit mass	e	dimensionless	$(\mathcal{L}/\mathcal{T})^2$	e/c^2
ʒular momentum of an orbit per unit mass	ℓ	$\mathcal{L}$	$\mathcal{L}^2/\mathcal{T}$	ℓ/c
Planck's constant	$\hbar$	$\mathcal{L}^2$	$\mathcal{M}(\mathcal{L}^2/\mathcal{T})$	$G\hbar/c^3$

APPENDIX B

Curvature Quantities

The following tables give useful quantities for the simplest of the geometries considered in the text. Specifically they give the metric, Christoffel symbols, Riemann curvature, and Einstein curvature. These are enough to form the geodesic equations and the Einstein equation.

Only nonzero components are shown, and only those nonzero components sufficient to construct the rest by summetries. For instance, we don't give both Γ^t_{tr} and Γ^t_{rt}, since $\Gamma^\alpha_{\beta\gamma}$ is symmetric in β and γ. Similarly other nonzero components of the Riemann curvature can be found from the ones displayed by making use of the symmetries in (21.29).

In each case one coordinate system is used and the Christoffel symbols are given in that coordinate basis. Both the Christoffel symbols and coordinate basis components of the curvature quantities can be computed using the *Mathematica* program *Curvature and the Einstein Equation* on the book website. However, curvature quantities are quoted in an orthonormal basis. This gives simpler expressions for highly symmetric metrics, and ones that are not singular at coordinate singularities. Since all the metrics considered are diagonal, we use an orthonormal basis whose vectors point along the coordinate directions. The coordinate components of these basis vectors are easily calculated from the metric according to the prescription in Example 7.9. Specifically,

$$(\mathbf{e}_{\hat{0}})^\alpha = [(-g_{00})^{-1/2}, 0, 0, 0],$$
$$(\mathbf{e}_{\hat{1}})^\alpha = [0, (g_{11})^{-1/2}, 0, 0], \quad \text{etc.} \qquad \begin{pmatrix}\text{diagonal}\\ \text{metrics}\end{pmatrix}$$

Components in this coordinate basis and the orthonormal basis are connected by (20.41), which in the case of the Einstein curvature reads

$$G_{\hat{\alpha}\hat{\beta}} = (e_{\hat{\alpha}})^\alpha \, (e_{\hat{\beta}})^\beta \, G_{\alpha\beta}$$

which, for these simple diagonal metrics, reduces to a simple prescription, e.g.,

$$G_{\hat{0}\hat{1}} = (-g_{00})^{-1/2} \, G_{01} (g_{11})^{-1/2} \text{, etc.} \qquad \text{(diagonal metrics).}$$

The analogous relation for the Riemann curvature is given in (21.25). Inverting these relations allows the coordinate basis components to be computed from the orthonormal basis components given.

The Ricci curvature components and the Ricci curvature scalar can be found from the Riemann curvature components in an orthonormal basis by

$$R_{\hat{\alpha}\hat{\beta}} = \eta^{\hat{\gamma}\hat{\delta}} R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\delta}}, \quad R = \eta^{\hat{\alpha}\hat{\beta}} R_{\hat{\alpha}\hat{\beta}}.$$

Schwarzschild Geometry

- **Metric (Schwarzschild coordinates):**

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- **Christoffel Symbols:**

$\Gamma^t_{tr} = (M/r^2)(1-2M/r)^{-1}$

$\Gamma^r_{tt} = (M/r^2)(1-2M/r)$

$\Gamma^r_{rr} = -(M/r^2)(1-2M/r)^{-1}$

$\Gamma^r_{\theta\theta} = -(r-2M)$

$\Gamma^r_{\phi\phi} = -(r-2M)\sin^2\theta$

$\Gamma^\theta_{r\theta} = 1/r$

$\Gamma^\theta_{\phi\phi} = -\cos\theta\,\sin\theta$

$\Gamma^\phi_{r\phi} = 1/r$

$\Gamma^\phi_{\theta\phi} = \cot\theta$

- **An Orthonormal Basis:**

$(e_{\hat{t}})^\alpha = \left[(1-2M/r)^{-1/2}, 0, 0, 0\right]$

$(e_{\hat{r}})^\alpha = \left[0, (1-2M/r)^{1/2}, 0, 0, 0\right]$

$(e_{\hat{\theta}})^\alpha = [0, 0, 1/r, 0]$

$(e_{\hat{\phi}})^\alpha = [0, 0, 0, 1/(r\sin\theta)]$

- **Riemann Curvature:**

$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -2M/r^3$

$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = +2M/r^3$

$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = +M/r^3$

$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -M/r^3$

- **Einstein Curvature**

$G_{\hat{\alpha}\hat{\beta}} = 0$

Spherically Symmetric Geometries

- **Metric:**

$$ds^2 = -e^{\nu(r,t)}dt^2 + e^{\lambda(r,t)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- **Christoffel Symbols:** (a prime denotes a partial derivative with respect to r; a dot denotes a partial derivative with respect to t)

$\Gamma^t_{tt} = \dot{\nu}/2$

$\Gamma^t_{tr} = \nu'/2$

$\Gamma^t_{rr} = e^{\lambda-\nu}\dot{\lambda}/2$

$\Gamma^r_{tt} = e^{\nu-\lambda}\nu'/2$

$\Gamma^r_{tr} = \dot{\lambda}/2$

$\Gamma^r_{rr} = \lambda'/2$

$\Gamma^r_{\theta\theta} = -e^{-\lambda}r$

$\Gamma^r_{\phi\phi} = -e^{-\lambda}r\sin^2\theta$

$\Gamma^\theta_{r\theta} = 1/r$

$\Gamma^\theta_{\phi\phi} = -\cos\theta\,\sin\theta$

$\Gamma^\phi_{r\phi} = 1/r$

$\Gamma^\phi_{\theta\phi} = \cot\theta$

- **An Orthonormal Basis:**
  $(e_{\hat{t}})^\alpha = \left[e^{-\nu/2}, 0, 0, 0\right]$
  $(e_{\hat{r}})^\alpha = \left[0, e^{-\lambda/2}, 0, 0\right]$
  $(e_{\hat{\theta}})^\alpha = [0, 0, 1/r, 0]$
  $(e_{\hat{\phi}})^\alpha = [0, 0, 0, 1/(r\sin\theta)]$

- **Riemann Curvature:**
  $R_{\hat{t}\hat{r}\hat{t}\hat{r}} = e^{-\lambda}\left[2\nu'' + (\nu')^2 - \lambda'\nu'\right]/4 - e^{-\nu}(2\ddot{\lambda} + \dot{\lambda}^2 - \dot{\nu}\dot{\lambda})/4$
  $R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = e^{-\lambda}\nu'/(2r)$
  $R_{\hat{t}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{r}\hat{\phi}} = e^{-(\nu+\lambda)/2}\dot{\lambda}/(2r)$
  $R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = e^{-\lambda}\lambda'/(2r)$
  $R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = (e^{-\lambda} - 1)/r^2$

- **Einstein Curvature:**
  $G_{\hat{t}\hat{t}} = e^{-\lambda}(-1 + e^{\lambda} + r\lambda')/r^2$
  $G_{\hat{t}\hat{r}} = e^{-(\nu+\lambda)/2}\dot{\lambda}/r$
  $G_{\hat{r}\hat{r}} = e^{-\lambda}(1 - e^{\lambda} + r\nu')/r^2$
  $G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = e^{-\lambda}\left[2\nu'' + (\nu')^2 + 2(\nu' - \lambda')/r - \nu'\lambda'\right]/4 - e^{-\nu}\left[2\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}\right]/4$

Friedman-Robertson-Walker (FRW) Geometries

- **Metric:**

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

- **Christoffel Symbols:**
  $\Gamma^t_{rr} = a\dot{a}/(1 - kr^2)$
  $\Gamma^t_{\theta\theta} = r^2 a\dot{a}$
  $\Gamma^t_{\phi\phi} = r^2\sin^2\theta\, a\dot{a}$
  $\Gamma^r_{rr} = kr/(1 - kr^2)$
  $\Gamma^r_{\theta\theta} = -r(1 - kr^2)$
  $\Gamma^r_{\phi\phi} = -r(1 - kr^2)\sin^2\theta$
  $\Gamma^r_{tr} = \dot{a}/a$
  $\Gamma^\theta_{r\theta} = 1/r$
  $\Gamma^\theta_{\phi\phi} = -\cos\theta\,\sin\theta$
  $\Gamma^\theta_{t\theta} = \dot{a}/a$
  $\Gamma^\phi_{r\phi} = 1/r$
  $\Gamma^\phi_{\theta\phi} = \cot\theta$
  $\Gamma^\phi_{t\phi} = \dot{a}/a$

- **An Orthonormal Basis:**
  $(e_{\hat{t}})^\alpha = [1, 0, 0, 0]$
  $(e_{\hat{r}})^\alpha = [0, \sqrt{1 - kr^2}, 0, 0]/a$
  $(e_{\hat{\theta}})^\alpha = [0, 0, 1/r, 0]/a$
  $(e_{\hat{\phi}})^\alpha = [0, 0, 0, 1/(r\sin\theta)]/a$

- **Riemann Curvature:**
  $R_{\hat{t}\hat{r}\hat{t}\hat{r}} = R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = -\ddot{a}/a$
  $R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = (k + \dot{a}^2)/a^2$

- **Einstein Curvature:**
  $G_{\hat{t}\hat{t}} = 3\left(k + \dot{a}^2\right)/a^2$
  $G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -\left(k + \dot{a}^2 + 2a\ddot{a}\right)/a^2$

Static, Weak Field Geometry

- **Metric:** $\Phi = \Phi(\vec{x}) = \Phi(x, y, z)$

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)(cdt)^2 + \left(1 - \frac{2\Phi}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right).$$

- **Christoffel Symbols:** (to linear order in Φ/c^2):

$$\Gamma^t_{tx} = \frac{1}{c^2}\frac{\partial\Phi}{\partial x}, \qquad \Gamma^x_{tt} = \frac{\partial\Phi}{\partial x},$$
$$\Gamma^x_{xx} = -\frac{1}{c^2}\frac{\partial\Phi}{\partial x}, \quad \Gamma^x_{xy} = -\frac{1}{c^2}\frac{\partial\Phi}{\partial y}, \quad \Gamma^x_{yy} = \frac{1}{c^2}\frac{\partial\Phi}{\partial x}$$

  plus cyclic permutations of (x, y, z).

- **An Orthonormal Basis:** (to leading order in $1/c$):

$$\left(e_{\hat{0}}\right)^\alpha = (1/c, 0, 0, 0), \quad \left(e_{\hat{1}}\right)^\alpha = (0, 1, 0, 0),$$
$$\left(e_{\hat{2}}\right)^\alpha = (0, 0, 1, 0), \quad \left(e_{\hat{3}}\right)^\alpha = (0, 0, 0, 1)$$

- **Riemann Curvature:** (to linear order in Φ/c^2):

$$R_{\hat{t}\hat{x}\hat{t}\hat{x}} = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial x^2}, \qquad R_{\hat{t}\hat{x}\hat{t}\hat{y}} = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial x\,\partial y},$$
$$R_{\hat{x}\hat{y}\hat{x}\hat{y}} = \frac{1}{c^2}\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right), \quad R_{\hat{x}\hat{y}\hat{x}\hat{z}} = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial y\,\partial z}$$

  plus cyclic permutations of (x, y, z).

- **Einstein Tensor:** (to linear order in Φ/c^2):

$$G_{\hat{t}\hat{t}} = \frac{2}{c^2}\nabla^2\Phi.$$

Linearized Gravity

- **Metric:**

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} + h_{\alpha\beta}(x)$$

- **Christoffel Symbols:**

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\,\eta^{\alpha\delta}\left(\frac{\partial h_{\delta\beta}}{\partial x^{\gamma}} + \frac{\partial h_{\delta\gamma}}{\partial x^{\beta}} - \frac{\partial h_{\beta\gamma}}{\partial x^{\delta}}\right)$$

- **An Orthonormal Basis:** (same as the coordinate basis to zeroth order)
- **Riemann Curvature:**

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial^2 h_{\alpha\delta}}{\partial x^{\beta}\partial x^{\gamma}} + \frac{\partial^2 h_{\beta\gamma}}{\partial x^{\alpha}\partial x^{\delta}} - \frac{\partial^2 h_{\alpha\gamma}}{\partial x^{\beta}\partial x^{\delta}} - \frac{\partial^2 h_{\beta\delta}}{\partial x^{\alpha}\partial x^{\gamma}}\right)$$

- **Einstein Curvature:**

$$G_{\alpha\beta} = \frac{1}{2}\left(-\Box\,\bar{h}_{\alpha\beta} + \frac{\partial V_{\alpha}}{\partial x^{\beta}} + \frac{\partial V_{\beta}}{\partial x^{\alpha}} - \eta_{\alpha\beta}\,\frac{\partial V^{\gamma}}{\partial x^{\gamma}}\right)$$

where $V_{\alpha} \equiv \partial\bar{h}^{\beta}_{\alpha}/\partial x^{\beta}$ with $\bar{h}^{\beta}_{\alpha} \equiv h^{\beta}_{\alpha} - (1/2)\delta^{\beta}_{\alpha}h^{\gamma}_{\gamma}$ and

$$\Box \equiv \eta^{\alpha\beta}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}} = -\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2.$$